JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 27, 338–366 (1969)

Dirichlet Problems for Mildly Nonlinear Elliptic Difference Equations*

G. T. McAllister

Center for the Application of Mathematics
Lehigh University, Bethlehem, Pennsylvania

Submitted by R. J. Duffin

Introduction

Let Ω be a rectangular region with sides of (rational) length σ_1 and σ_2, where the side of length σ_1 is on the x-axis. We denote the sides of Ω by s_i, $i = 1,\ldots, 4$, with s_1 on the line $x = \sigma_1$ and the ordering proceeding in a counter-clockwise manner.

Place a square net on the plane and let Ω_h be those mesh points in Ω and let $\partial\Omega_h$ be those mesh points on the $\partial\Omega$. We denote by $\Omega_{h(x)}$, $\Omega_{h(y)}$, etc. those subsets of $\bar{\Omega}_h = \Omega_h + \partial\Omega_h$ on which x-difference quotients, y-difference quotients, etc. are defined. If $V(P)$ is a mesh function, then $V_x(P)$ is the forward difference quotient in the x-direction, $V_{\bar{x}}(P)$ is the backward difference quotient in the x-direction, etc. If $v(P)$ is a continuously differentiable function, then $v_{,x}(P) = \partial v(P)/\partial x$, $v_{,xy} = \partial^2 v(P)/\partial x\, \partial y$, etc.

In every section we shall denote some constants, which we explicitly compute, by A_i with i assuming positive integer values. For any function $V(P)$ we define $[V(P)]$ to be the max $|V(P)|$ with P going over the domain of definition of $V(P)$. Each section of this paper is divided into parts which are labeled with Roman numerals.

In Sections 1 and 2, we establish conditions of uniqueness and constructive existence for the problem

$$\begin{aligned} a(P)\, U_{x\bar{x}}(P) + c(P)\, U_{y\bar{y}}(P) &= f\{P, U(P), U_x(P), U_y(P)\}, \; P \in \Omega_h \\ U(P) &= z(P), \; P \in \partial\Omega_h, \end{aligned} \tag{*}$$

where we assume that positive constants λ and L exist such that for every $P \in \Omega_h$,

$$0 < \lambda \leqslant \{a(P), c(P)\} \leqslant L \tag{**}$$

* Research supported by Contract GP-7192 with the National Science Foundation.

and that a positive constant, $L(f)$, exists such that for each $P \in \Omega$ and for any everywhere finite mesh function $V(P)$ we have that

$$|f(P, V(P), V_x(P), V_y(P))| \leqslant L(f).$$

This problem is considered when we have:

(i) $a(P)$, $c(P)$ Hölder continuous over $\bar{\Omega}$ and satisfy (**);

(ii) $a(P)$, $c(P)$ are single-valued functions which only satisfy (**).

Under the condition (i) we need only to assume that $z(P)$ has uniformly founded second-order difference quotients and we apply results in [1]. For condition (ii), treated in Section 2, we obtain bounds on first-order difference quotients assuming that $z(P)$ is analytic on each side s_i, $i = 1,..., 4$, of the $\partial\Omega$ and we apply these results to (*). Numerous examples are given in Section 1 which do not satisfy the conditions of our theorems, in particular the non-existence of $L(f)$, and these examples are solved under various other restrictions. Our bounds on first-order difference quotients under condition (ii), in Section 2, have immediate applications to existence theorems to quasilinear elliptic difference Dirichlet problems. The existence and uniqueness criteria of these sections are new and treat problems which are not discussed in [2]. The question of convergence of *the* discrete solution to the continuous solution is treated exactly as in [2].

In Section 3 we obtain explicit bounds on some absolute constants involved in discrete sum inequalities which are used in this paper.

1. Mildly Nonlinear Problems with Hölder Continuous Coefficients

In this section we shall show how *a priori* bounds may be used in obtaining existence—constructive existence—and uniqueness theorems for a mildly nonlinear Dirichlet problem.

(I) *Constant coefficients*

In this part we seek a mesh function $U(P)$ such that

$$\begin{aligned} \Delta_h U(P) &= f(P, U(P), U_x(P), U_y(P)), \ P \in \Omega_h \\ U(P) &= 0, \ P \in \partial\Omega_h . \end{aligned} \tag{1}$$

Let $\xi(P)$ be any everywhere finite mesh function defined on $\bar{\Omega}_h$. Let $\mathcal{O}\!\ell_1 = \{\xi(P) : \xi(P) = 0, \ P \in \partial\Omega_h\}$. Let us assume that for *every* $\xi(P) \in \mathcal{O}\!\ell_1$ there exists a positive constant $L(f)$, which is independent of $\xi(P)$, such that

$$|f(P, \xi(P), \xi_x(P), \xi_y(P))| \leqslant L(f). \tag{2}$$

Let $\psi(\xi(\cdot); P)$ be *the* solution to the problem

$$\begin{aligned} \Delta_h \psi(\xi(\cdot); P) &= f(P, \xi(P), \xi_x(P), \xi_y(P)), \ P \in \Omega_h, \\ \psi(\xi(\cdot); P) &= 0, \ P \in \partial\Omega_h. \end{aligned} \tag{3}$$

Then there exists a positive constant A_1 such that for each $\xi(P) \in \mathcal{O}\!\mathcal{l}_1$ we have the estimate

$$[\psi(\xi(\cdot); P)] \leqslant A_1 L(f); \tag{4}$$

in fact it has been shown that

$$A_1 \leqslant d^2/8, \tag{5}$$

where $d = \min\{\sigma_1, \sigma_2\}$ and, σ_1 and σ_2 the lengths of the sides of Ω. Let $\mathcal{O}\!\mathcal{l}_2 = \{\xi(P) : \xi(P) \in \mathcal{O}\!\mathcal{l}_1, [\xi(P)] \leqslant A_1 L(f)\}$. Then $\psi(\xi(\cdot); \cdot)$ is a mapping from $\mathcal{O}\!\mathcal{l}_1$ into $\mathcal{O}\!\mathcal{l}_2$, i.e.,

$$\psi(\xi(\cdot); \cdot) : \mathcal{O}\!\mathcal{l}_1 \to \mathcal{O}\!\mathcal{l}_2.$$

Let us define the positive number A_2 by the relation

$$A_2 = 4\gamma_1(\eta) \iint \rho^{-\eta}\, dx\, dy, \tag{6}$$

where $\eta \in (0, 1)$, $\gamma_1(\eta)$ is determined in McAllister [1], and the double integral in (6) is over the disc which circumscribes the region Ω. Application of *Remark* 2 of Section 2 part (b) of McAllister [1] produces the bound

$$[[\psi_x(\xi(\cdot); P)], [\psi_y(\xi(\cdot); P)]] \leqslant A_2 L(f) \tag{7}$$

for any $\xi(P) \in \mathcal{O}\!\mathcal{l}_1$ and where the difference quotients of ψ are taken with respect to the components of P. Let us define the set

$$\mathcal{O}\!\mathcal{l}_2' = \{\xi(P) : \xi(P) \in \mathcal{O}\!\mathcal{l}_1, [[\xi_x(P)], [\xi_y(P)]] \leqslant A_2 L(f)\}.$$

Then we have that $\psi(\xi(\cdot); \cdot)$ is a mapping from $\mathcal{O}\!\mathcal{l}_1$ into $\mathcal{O}\!\mathcal{l}_2'$, i.e.,

$$\psi(\xi(\cdot); \cdot) : \mathcal{O}\!\mathcal{l}_1 \to \mathcal{O}\!\mathcal{l}_2'.$$

The functions which are solutions to (1) are precisely the fixed points of the mapping $\psi(\xi(\cdot); P)$ i.e. those mesh functions $\xi_0(P)$ such that

$$\psi(\xi_0(\cdot); P) = \xi_0(P)$$

for $P \in \bar{\Omega}_h$. Since ψ maps $\mathcal{O}\!\mathcal{l}_1$ into $\mathcal{O}\!\mathcal{l}_2'$ we may restrict our search for fixed-points to the set $\mathcal{O}\!\mathcal{l}_2'$.

We shall now prove the following theorem.

THEOREM 1.1. *If* $\xi_1(P)$, $\xi_2(P) \in \mathcal{O}\!\ell_2'$ *and if* $f(P, \xi(P), \xi_x(P), \xi_y(P))$ *is continuously differentiable with respect to the arguments* $\xi(P)$, $\xi_x(P)$ *and* $\xi_y(P)$, *then*

$$\langle \psi(\xi_1(\,\cdot\,); P) - \psi(\xi_2(\,\cdot\,); P)\rangle \leqslant A_3\langle \xi_1(P) - \xi_2(P)\rangle, \tag{8}$$

where

$$\langle \xi(P)\rangle = [\xi(P)] + [\xi_x(P)] + [\xi_y(P)], \tag{9}$$

and

$$A_3 \leqslant [A_1\,, 2A_2] \cdot [[f_{,\xi}], [f_{,p}], [f_{,q}]];$$

here $f_{,\xi} = \partial f/\partial \xi, f_{,p} = \partial f/\partial \xi_x$, *and* $f_{,q} = \partial f/\partial \xi_y$.

PROOF. We have that $\psi(\xi_1(\,\cdot\,); P) - \psi(\xi_2(\,\cdot\,); P)$ solves the problem

$$\begin{aligned} \Delta_h(\psi(\xi_1(\,\cdot\,); P) - \psi\,(\xi_2(\,\cdot\,); P)) &= f(P, \xi_1(P), \xi_{1x}(P), \xi_{1y}(P)) \\ &\quad - f(P, \xi_2(P), \xi_{2x}(P), \xi_{2y}(P)),\ P \in \Omega_h \quad (10) \\ \psi(\xi_1(\,\cdot\,); P) - \psi(\xi_2(\,\cdot\,); P) = 0, &\qquad P \in \partial\Omega_h\,. \end{aligned}$$

Therefore,

$$\begin{aligned} &[\psi(\xi_1\,\cdot\,); P) - \psi(\xi_2(\,\cdot\,); P)] \\ &\quad \leqslant A_1\{[f_{,\xi}] \cdot [\xi_1 - \xi_2] + [f_{,p}] \cdot [\xi_{1x} - \xi_{2x}] + [f_{,q}] \cdot [\xi_{1y} - \xi_{2y}]\}. \end{aligned} \tag{11}$$

Now apply to (10) the remark we used to define $\mathcal{O}\!\ell_2'$ to get

$$\begin{aligned} &[[\psi_x(\xi_1(\,\cdot\,); P) - \psi_x(\xi_2(\,\cdot\,); P)], [\psi_y(\xi_1(\,\cdot\,); P) - \psi_y(\xi_2(\,\cdot\,); P)]] \\ &\quad \leqslant A_2\{[f_{,\xi}] \cdot [\xi_1 - \xi_2] + [f_{,p}] \cdot [\xi_{1x} - \xi_{2x}] + [f_{,q}] \cdot [\xi_{1y} - \xi_{2y}]\}. \end{aligned} \tag{12}$$

Combining (11) and (12) concludes the proof.

As an immediate consequence of the *Banach Fixed point Theorem* we have the following corollary.

COROLLARY 1.1. *If* $A_3 < 1$, *then there exists a unique solution to the problem given in* (1).

REMARK 1.1. The constraints imposed by Bers on $f_{,U}$, $f_{,p}$ and $f_{,q}$ are satisfied when $f(P, \xi, \xi_x, \xi_y) \equiv \xi^3 + \sin(\xi_x + \xi_y)$ with

$$f_\xi = 3\xi^2, [[f_{,p}], [f_{,q}]] \leqslant 1,$$

but our conditions are not satisfied. However, if

$$f(P, \xi, \xi_x, \xi_y) \equiv \sin(\xi^2 + \xi_x^2 + \xi_y^2),$$

then our conditions are satisfied but those of Bers are not satisfied; here

$$f_{,\xi} = 2\xi \cos(\xi^2 + \xi_x^2 + \xi_y^2), f_{,p} = 2\xi_x \cos(\xi^2 + \xi_x^2 + \xi_y^2)$$

and $f_{,q} = 2\xi_y \cos(\xi^2 + \xi_x^2 + \xi_y^2)$ are not bounded unless we have *a priori* bounds, i.e., unless we may assume that the problem has a classical solution with at least three continuous derivatives over $\bar{\Omega}$. Then we could use the result on the discretization error to obtain bounds on $f_{,p}$ and $f_{,q}$ but such a result would not allow us to obtain *Theorem* 1.1 and does beg the question.

(II) *Some Examples not Covered by Our Theorem*

In this part we will consider specific problems which do not satisfy the conditions of Theorem 4.1. These problems will be treated by modifying the method of proof of that theorem.

(A) Let $U(P)$ be a solution to the problem

$$\begin{aligned} \Delta_h U(P) &= \mu U(P), \qquad P \in \Omega_h, \\ U(P) &= 0, \qquad P \in \partial\Omega_h, \end{aligned} \tag{13}$$

where μ is some constant. If $\mu \geqslant 0$, then $U(P) \equiv 0$ for all $P \in \bar{\Omega}_h$. If $\mu < 0$, then we shall obtain bounds on difference quotients of $U(P)$ in terms of $[U(P)]$.

Taking difference quotients of the sum formula for $U(P)$ gives

$$\begin{aligned} | U_x(P) | &= | \mu | \cdot \Big| h^2 \sum_{Q \in \Omega_h} G_x(P; Q) U(Q) \Big| \\ &\leqslant 4 | \mu | \cdot [U(P)] \cdot \gamma_1(\eta)\, d^{1-\eta}/(1-\eta), \end{aligned} \tag{14}$$

where $\gamma_1(\eta)$ is given in [1], $\eta \in (0, 1)$, and d is the radius of the circumscribed circle of Ω. The same inequality holds for $U_y(P)$.

In a similar fashion

$$\begin{aligned} | U_{x\bar{x}}(P) | &= | \mu | \cdot \Big| h^2 \sum_{Q \in \Omega_h} G_{x\bar{x}}(P; Q)\, U(Q) \Big| \\ &\leqslant | \mu | \cdot h^2 \sum_{Q \in \Omega_h} | G_{x\bar{x}}(P; Q)(U(Q) - U(P)) | \\ &\quad + | U(P) | \cdot | \mu | \Big| h^2 \sum_{Q \in \Omega_h} G_{x\bar{x}}(P; Q) \Big| \\ &\leqslant [U(P)] \{ | \mu |^2\, 32 \gamma_2(\eta)\, \gamma_1(\eta) (d^{1-\eta}/(1-\eta))^2 + | \mu | \cdot \gamma_5 \}, \end{aligned} \tag{15}$$

where $\gamma_2(\eta)$ and γ_5 are given in [1]. Therefore,

$$\{[U_{x\bar{x}}(P)], [U_{y\bar{y}}(P)]\} \leqslant |\mu|^2 [U(P)]\{32\gamma_2(\eta)\gamma_1(\eta)(d^{1-1}/(1-\eta))^2 + \gamma_5/|\mu|\}. \tag{16}$$

Now note that a result of Nitsche and Nitsche [3, p. 296] gives

$$\| U_{x\bar{x}} \|^2 + 2\| U_{xy} \|^2 + \| U_{y\bar{y}} \|^2 \leqslant 2|\mu|^2 \cdot \pi d^2 [U(P)]^2 \leqslant 4\pi d^2 A_4 |\mu|^2 \{\| U_{x\bar{x}} \|^2 + \| U_{y\bar{y}} \|^2\}, \tag{17}$$

where we have also used the discrete *Sobolev Inequality* [3, p. 301] and A_4 is the positive constant, depending on Ω, which is associated with that inequality (see section 3 for an explicit bound on A_4; if Ω is the unit square then $A_4 = \sqrt{1+\pi}/4$). Therefore, if $U(P)$ is not to be identically zero, then we must have

$$|\mu| \geqslant 1/(4\pi d^2 A_4)^{1/2};$$

hence the bound in (16) has no singularities.

The results in (16) and (14) are related to discrete analogues of some of the results in Smolickii [5; p. 205].

We also note, with reference to problem (13), that the condition $h^2 \sum_{\Omega_h} U^2(P)$ bounded implies that $U(P)$ is bounded.

(B) Now we shall investigate solutions to the problem

$$\begin{aligned} \Delta_h U(P) &= \mu(U_x^2 + U_y^2)^{1/m}, \qquad P \in \Omega_h, \\ U(P) &= 0, \qquad P \in \partial\Omega_h, \end{aligned} \tag{18}$$

where $m \geqslant 2$ and μ is a constant; only in the case $m = 2$ are the uniqueness conditions of L. Bers satisfied. The *Schwarz Inequality* gives

$$\begin{aligned} |U_x(P)| &= |\mu| \cdot \left| h^2 \sum_{Q\in\Omega_h} G_x(P;Q)(U_x^2 + U_y^2)^{1/m} \right| \\ &\leqslant |\mu| \cdot \gamma_1(\eta) h^2 \sum_{Q\in\Omega_h} \rho_{PQ}^{-1-\eta}(U_x^2 + U_y^2)^{1/m} \\ &\leqslant |\mu| \cdot \gamma_1(\eta)\Big(h^2 \sum_{Q\in\Omega_h} \rho_{PQ}^{-2-2\eta+2\alpha}\Big)^{1/2} \Big(h^2 \sum_{Q\in\Omega_h} \rho_{PQ}^{-2\alpha}(U_x^2 + U_y^2)^{2/m}\Big)^{1/2} \\ &\leqslant 4|\mu|\gamma_1(\eta) d^{\alpha-\eta} \Big(h^2 \sum_{Q\in\Omega_h} \rho_{PQ}^{-2\alpha}(U_x^2 + U_y^2)^{2/m}\Big)^{1/2} \Big/ \sqrt{2(\alpha-\eta)}, \end{aligned} \tag{19}$$

where d is the diameter of the circumscribed circle to Ω, $\eta \in (0, 1)$, and

$\alpha > \eta$. The same bound holds when we consider $U_y(P)$. We shall now bound the sum in (19).

Applying (17) gives

$$\| U_{x\bar{x}} \|^2 + 2\| U_{xy} \|^2 + \| U_{y\bar{y}} \|^2 \leqslant 2\mu^2 \|(U_x^2 + U_y^2)^{1/m} \|^2. \tag{20}$$

Therefore the discrete *Gauss Theorem* gives

$$h^2 \sum U_x^4 \leqslant 3h^2 \sum | U_x^2 U_{xx} U | \leqslant 3[U]\Big(h^2 \sum_{\Omega_h} U_x^4\Big)^{1/2} \Big(h^2 \sum_{\Omega_h} U_{x\bar{x}}^2\Big)^{1/2}$$

with a similar result holding for U_y. Therefore,

$$h^2 \sum_{\Omega_h} \{U_x^4 + U_y^4\} \leqslant 18 A_4 \{\| U_{x\bar{x}} \|^2 + \| U_{y\bar{y}} \|^2\}^2. \tag{20$'$}$$

Now returning to (19) we have

$$\begin{aligned} h^2 \sum_{\Omega_h} \rho_{PQ}^{-2\alpha} (U_x^2 + U_y^2)^{2/m} &\leqslant \Big(h^2 \sum_{\Omega_h} \rho_{PQ}^{-4\alpha}\Big)^{1/2} \Big(h^2 \sum_{\Omega_h} (U_x^2 + U_y^2)^{4/m}\Big)^{1/2} \\ &\leqslant (d^{1-4\alpha}/(1-4\alpha))^{1/2} \Big\{h^2 \sum_{\Omega_h} (U_x^4 + U_y^4) + d^2\pi\Big\}^{1/2} \qquad (21) \\ &\leqslant (d^{1-4\alpha}/(1-4\alpha))^{1/2} \{\sqrt{18A_4}(\| U_{x\bar{x}} \|^2 + \| U_{y\bar{y}} \|^2) + d\sqrt{\pi}\}, \end{aligned}$$

where we are assuming $1/4 > \alpha > \eta > 0$ and where we are using the result that $a^{8/m} \leqslant a^4$ if $a \geqslant 1$ and $a^{8/m} \leqslant 1$ if $a \in [0, 1]$. Hence,

$$\begin{aligned} | U_x(P)| \leqslant 4 \,| \mu |\, \gamma_1(\eta)\, d^{\alpha-\eta+(1-4\alpha)/4} \{(18A_4)^{1/4} (\| U_{x\bar{x}} \| + \| U_{y\bar{y}} \|) \\ + d^{1/2}\pi^{1/4}\}/\sqrt{2(\alpha-\eta)\sqrt{1-4\alpha}}. \qquad (22) \end{aligned}$$

Combining the inequality used in (21) to (20) gives

$$\begin{aligned} \| U_{x\bar{x}} \|^2 + \| U_{y\bar{y}} \|^2 &\leqslant 2 \,| \mu |\{\| U_x \|^2 + \| U_y \|^2\} + 2d^2\pi \,| \mu | \\ &\leqslant 2 \,| \mu |\, \sqrt{\pi}\, d[U]\{\| U_{x\bar{x}} \| + \| U_{y\bar{y}} \|\} + 2d^2\pi \,| \mu | \qquad (23) \\ &\leqslant 4 \,| \mu |\, \sqrt{\pi}\, dA_4\{\| U_{x\bar{x}} \|^2 + \| U_{y\bar{y}} \|^2\} + 2d^2\pi \,| \mu |. \end{aligned}$$

Now assume $4 \,| \mu |\, \sqrt{\pi}\, dA_4 < 1$. Then

$$\| U_{x\bar{x}} \|^2 + \| U_{y\bar{y}} \|^2 \leqslant 2d^2\pi \,| \mu |/(1 - 4 \,| \mu |\, \sqrt{\pi}\, dA_4). \tag{24}$$

Therefore,

$$[U] \leqslant 2A_4 d^2\pi \,| \mu |/(1 - 4 \,| \mu |\, \sqrt{\pi}\, dA_4) \tag{25}$$

and

$$[[U_x], [U_y]] \leqslant 4 \mid \mu \mid \gamma_1(\eta)\, d^{\alpha-\eta+(1-4\alpha)/4}\{(18A_4)^{1/4} \cdot (1 + 2d^2\pi \mid \mu \mid)/(1 - 4d \mid \mu \mid \sqrt{\pi}\, A_4) + d^{1/2}\pi^{1/4}\}/\sqrt{2(\alpha - \eta)}\, \sqrt{1 - 4\alpha}.$$

In the case that $m = 2$ the term $d\sqrt{\pi}$ does not appear in (21) and in the case that $4 \mid \mu \mid \sqrt{\pi}\; dA_4 < 1$ we have $U_{x\bar{x}} = 0$, $U_{y\bar{y}} = 0$ and $U_{xy} = 0$. Therefore $U_x = U_y = 0$ and hence $U \equiv 0$. In the case $m > 2$ we do not obtain a bound on f_p or f_q and hence we can't apply uniqueness. We observe however, that if we use the estimate

$$\begin{aligned} h^2 \sum_{\Omega_h} (U_x^2 + U_y^2)^{2/m} &\leqslant \left(h^2 \sum_{\Omega_h} (U_x^2 + U_y^2)\right)^{2/m} \left(h^2 \sum_{\Omega_h}\right)^{1-2/m} \\ &\leqslant (\pi d^2)^{1-2/m} (\sqrt{\pi}\, d[U]\{\| U_{x\bar{x}} \| + 2 \| U_{xy} \| + \| U_{y\bar{y}} \|\})^{2/m} \\ &\leqslant (\pi d^2)^{1-2/m} (2\sqrt{\pi}\, dA_4\{\| U_{x\bar{x}} \|^2 + 2 \| U_{xy} \|^2 + \| U_{y\bar{y}} \|^2\})^{2/m}, \end{aligned} \tag{26}$$

applying (26) to (20) gives

$$\begin{aligned} \| U_{x\bar{x}} \|^2 + 2 \| U_{xy} \|^2 + \| U_{y\bar{y}} \|^2 &\leqslant 2 \mid \mu \mid^2 \|(U_x^2 + U_y^2)^{1/m} \|^2 \\ &\leqslant 2 \mid \mu \mid^2 \{(\pi d^2)^{1-2/m} (2\sqrt{\pi}\, dA_4\{\| U_{x\bar{x}} \|^2 + 2 \| U_{xy} \|^2 + \| U_{y\bar{y}} \|^2\})^{2/m}\} \end{aligned} \tag{27}$$

Then from (27) we obtain the bound

$$\begin{aligned} &\| U_{x\bar{x}} \|^2 + 2 \| U_{xy} \|^2 + \| U_{y\bar{y}} \|^2 \\ &\qquad \leqslant (2 \mid \mu \mid^2 (\pi d^2)^{1-2/m} (2\sqrt{\pi}\, dA_4)^{2/m})^{m/(m-2)} \end{aligned} \tag{28}$$

and hence

$$[U] \leqslant (2A_4^2(2 \mid \mu \mid^2 (\pi d^2)^{1-2/m} (2\sqrt{\pi}\, dA_4)^{2/m})^{m/(m-2)})^{1/2}. \tag{29}$$

We now substitute (29) into (23) to get a bound *independent* of the size of $4 \mid \mu \mid \sqrt{\pi}\, dA_4$. More precisely we obtain the bound

$$\begin{aligned} &h^2 \sum_{Q\in\Omega_h} \rho_{PQ}^{-2\alpha}(U_x^2 + U_y^2)^{2/m} \\ &\qquad \leqslant 2 \left(h^2 \sum_{Q\in\Omega_h} \rho_{PQ}^{-2\alpha m/(m-1)}\right)^{(m-1)/m} \left(h^2 \sum_{Q\in\Omega} (U_x^4 + U_y^4)\right)^{1/m}; \end{aligned} \tag{30}$$

now apply (20)′ and (29) to (30) and plug the result into (19) to get a bound on $|U_x(P)|$ and $|U_y(P)|$ which is independent of the size of $4|\mu|\sqrt{\pi}\, dA_4$.

If we assume that $2|\mu|^2(\sqrt{\pi}\, d)^{(2m-4)/m}(2\sqrt{\pi}\, dA_4)^{2/m} < 1$ and if we assume that $\| U_{x\bar{x}} \| \geqslant 1$ and $\| U_{y\bar{y}} \| \geqslant 1$, then the estimate in (27) implies that $U_{x\bar{x}} = U_{y\bar{y}} = U_{xy} = 0$ and hence $U(P) \equiv 0$ for $P \in \bar{\Omega}_h$.

We note that for the problem posed in (18) the quantities $|f_{,p}|$ and $|f_{,q}|$ may be undefined, or at least arbitrarily large unless we are assured that $U_x^2 + U_y^2 \geqslant$ some positive constant which is independent of h. Hence, although we may bound $U(P)$ and its' first-order difference quotients we may not use *Theorem* 1.1 for this problem. If we were to consider the problem with $f = \mu(1 + U_x^2 + U_y^2)^{1/m}$, then the situation is altered considerably.

(C) Let us consider the problem given by

$$\begin{aligned} \Delta_h U(P) &= U(P)\, U_x(P), \qquad P \in \Omega_h, \\ U(P) &= 0, \qquad P \in \partial\Omega_h. \end{aligned} \tag{31}$$

The uniqueness conditions of Bers are not *a priori* satisfied and the methods presented above fail. Let $\mathfrak{A}_\nu = \{\xi(P) : [\xi(P)] \leqslant \nu\}$ with $\nu \geqslant 0$. Let $\psi(\xi(\cdot); P)$ be *the* solution to the problem

$$\begin{aligned} \Delta_h \psi(\xi(\cdot); P) &= \xi(P)\, \psi_x(\xi(\cdot); P), \qquad P \in \Omega_h, \\ \psi(\xi(\cdot); P) &= 0, \qquad P \in \partial\Omega_h, \end{aligned} \tag{32}$$

for $\xi(P) \in \mathfrak{A}_\nu$. Then we have the result that for each $\xi(P)$,

$$\psi(\xi(\cdot); P) \equiv 0, \qquad P \in \bar{\Omega}_h.$$

Hence ψ maps $\mathfrak{A}_\nu$ onto $\mathfrak{A}_0$ for each $\nu \geqslant 0$; i.e., each fixed-point of ψ, which is equivalent to each bounded solution of (31), must be zero. Therefore we have existence and uniqueness in this case whenever $\nu h \leqslant 1$; $\nu h > 1$ would correspond to a singular solution.

Now consider (31) under the boundary condition that $U(P) = z(P)$ for $P \in \partial\Omega_h$, where $z(P)$ is continuous on the $\partial\Omega_h$ and $z(P)$ has second-order difference quotients uniformly bounded on each side of $\partial\Omega_h$. Let

$$\mathfrak{A}_\nu' = \{\xi(P) : \xi(P) = z(P) \quad \text{for} \quad P \in \partial\Omega_h, \quad [\xi] \leqslant \nu\}.$$

Let $\psi(\xi(\cdot); P)$ be *the* solution to (32) for $\xi(P) \in \mathfrak{A}_\nu'$ but now $\psi = z(P)$ on the $\partial\Omega_h$. Let $H(P)$ be the discrete harmonic function associated with the boundary data $z(P)$. Then if $\nu h \leqslant 1$, Bers [2; p. 230],

$$[\psi] \leqslant [z].$$

Now seek those values of ν such that

$$[z] \leqslant \nu$$

with $h\nu < 1$. If ν' satisfies this inequality, then we have that ψ maps $\mathcal{O}\!\mathcal{l}'_{\nu'}$ into itself and since ψ is continuous in ξ it has at least one fixed-point in $\mathcal{O}\!\mathcal{l}'_{\nu'}$. Let $\xi_1(P)$, $\xi_2(P) \in \mathcal{O}\!\mathcal{l}'_{\nu'}$. Then

$$\begin{aligned} \Delta_h(\psi(\xi_1(\,\cdot\,); P) - \psi(\xi_2(\,\cdot\,); P)) - \xi_1(P)(\psi(\xi_1(\,\cdot\,); P) - \psi(\xi_2(\,\cdot\,); P))_x \\ = (\xi_1(P) - \xi_2(P))\, \psi_x(\xi_1(\,\cdot\,); P), \qquad P \in \Omega_h\,, \\ \psi(\xi_1(\,\cdot\,); P) - \psi(\xi_2(\,\cdot\,); P) = 0, \qquad P \in \partial\Omega_h\,. \end{aligned}$$

Now assume that $\nu'\gamma_1(\eta)\, d^{1-\eta}/(1-\eta) < 1$ for $\eta \in (0, 1)$. Then

$$[\psi_x(\xi(\,\cdot\,); P)] \leqslant \gamma_1(\eta)\, d^{1-\eta}[H_x]/(1 - \eta - \nu'\gamma_1(\eta)\, d^{1-\eta}). \tag{33}$$

Therefore

$$[\psi(\xi_1(\,\cdot\,); P) - \psi(\xi_2(\,\cdot\,); P)] \leqslant A_6[\xi_1(P) - \xi_2(P)],$$

where A_6 is A_5 times the right-hand side of the inequality in (33) and A_5 is given by the relation

$$\begin{aligned} &A_5 \leqslant d^2/4(2 - \nu d) \qquad \text{if} \quad \nu d < 2 \quad \text{and} \\ &A_5 \leqslant 2\theta \exp\{(\nu d \tanh^{-1}\Theta/\theta)\}/\nu^2(1-\theta)\tanh^{-1}\Theta, \qquad \text{if} \quad \nu d \geqslant 2, \end{aligned}$$

with $\Theta = (3\theta - \theta^2)/^2$ and $\theta \in (0, 1)$ such that $\nu h < 2\theta$. If $A_6 < 1$, then we have a constructive solution to the problem over $\mathcal{O}\!\mathcal{l}'_{\nu'}$. The case where we do not assume that quantity less than one is treated in the next example. Our treatment here is independent of an ℓ_p theory.

Many variations of the above procedures will produce regions $\mathcal{O}\!\mathcal{l}'_{\nu'}$ over which there is a unique fixed-point to problems of the type given by (31).

(III) *Systems with constant coefficients*

Let $U(P) = (U_1(P), \ldots, U_n(P))$ be a solution to the system

$$\begin{aligned} \Delta_h U_i(P) = f_i(P, U(P), U_x(P), U_y(P)), \qquad P \in \Omega_h\,, \\ U_i(P) = z_i(P), \qquad P \in \partial\Omega_h\,, \end{aligned} \tag{34}$$

where $i = 1, \ldots, n$. We may now restate our *Theorem* 1.1 for the system in (34) as follows: If we define $L(f) \equiv [L(f_1), \ldots, L(f_n)]$ and if this is uniformly bounded for all $\xi(P) \equiv (\xi_1(P), \ldots, \xi_n(P))$, then the function

$$\psi(\xi(\,\cdot\,); P) \equiv (\psi_1(\xi(\,\cdot\,); P), \ldots, \psi_n(\xi(\,\cdot\,); P))$$

satisfies the estimate

$$\langle \psi(\xi_1(\,\cdot\,); P) - \psi(\xi_2(\,\cdot\,); P)\rangle \leqslant A_7 \langle \xi_1(P) - \xi_2(P)\rangle, \tag{35}$$

where

$$\begin{aligned}
&\xi_1\,, \xi_2 \in \mathcal{O}\!\ell_2' \\
&\quad = \{\xi(P) : [\xi(P)] \leqslant A_1 L(f) + [z], [[\xi_x(P)], \xi_y(P)]] \leqslant A_2 L(f) + 2[z']\} \\
&\qquad\qquad [\xi(P)] \equiv \max_i\, [\xi_i(P)], [\xi_x] \equiv \max_i\, [\xi_{ix}(P)], \\
&\qquad\qquad\quad [z] \equiv \max_i\, [z_i], [f,_\xi] \equiv \max_i\, [f,_{\xi i}], \text{ etc.,} \\
&\qquad\qquad\qquad A_7 = [A_1\,, A_2] \cdot [[f,_\xi], [f,_p], [f,_q]],
\end{aligned}$$

and

$$\langle \xi_1(P) - \xi_2(P)\rangle = \max_i \langle \xi_{1i}(P) - \xi_{2i}(P)\rangle.$$

As an example of a system not covered by the above theorem we consider the problem, in part related to the Navier-Stokes equations,

$$\begin{aligned}
&\Delta_h U = UU_x + VU_y \\
&\Delta_h V = UV_x + VV_y \qquad\qquad P \in \Omega_h\,, \\
&U(P) = z_1(P), V(P) = z_2(P), \qquad P \in \partial\Omega_h\,.
\end{aligned} \tag{36}$$

Let $\mathcal{O}\!\ell_\nu = \{\xi(P) : \xi(P) = (\xi_1(P), \xi_2(P)), \xi_1(P) = z_1(P)$ and $\xi_2(P) = z_2(P)$ for $P \in \partial\Omega_h\,, [\xi] \leqslant \nu\}$. Let $\psi(\xi; P) = (\psi_1(\xi; P), \psi_2(\xi; P))$ where

$$\begin{aligned}
\Delta_h \psi_1 &= \xi_1 \psi_{1x} + \xi_2 \psi_{1y} \\
\Delta_h \psi_2 &= \xi_1 \psi_{2x} + \xi_2 \psi_{2y} \\
\psi_1 = z_1\,, &\qquad \psi_2 = z_2\,;
\end{aligned} \tag{37}$$

note that the system in (37) is uncoupled while that in (36) is coupled. Therefore, if $\nu h < 1$,

$$\begin{aligned}
[\psi] &\leqslant [H_1] \\
[\psi_2] &\leqslant [H_2],
\end{aligned} \tag{38}$$

where H_i, $(i = 1, 2)$, is the discrete harmonic function taking on the values z_i, $(i = 1, 2)$, on the $\partial\Omega_h$. Again, as in (33), we seek those values of ν such that

$$\max\{[H_1], [H_2]\} \leqslant \nu. \tag{39}$$

Then, for those values of ν, ψ maps $\mathcal{O}\!\ell_\nu$ into itself. Now we seek fixed points of ψ, i.e., we seek those ξ_0 such that

$$(\psi_1(\xi_0\,;P),\psi_2(\xi_0\,;P)) = \psi(\xi_0\,;P) = \xi_0(P) = (\xi_{01}(P), \xi_{02}(P)).$$

These fixed points are the only elements of $\mathcal{O}\!\ell_\nu$ which solve (36) in the set $\mathcal{O}\!\ell_\nu$. That such $\xi_0(P)$ exist is again assured by continuity. Let $\xi_1, \xi_2 \in \mathcal{O}\!\ell_\nu$. Then

$$\begin{aligned}
&\Delta_h(\psi_1(\xi_1\,;P) - \psi_1(\xi_2\,;P))\\
&\quad = (\xi_{11} - \xi_{21})\,\psi_{1x}(\xi_1\,;P) + (\xi_{12} - \xi_{22})\,\psi_{1y}(\xi_1\,;P)\\
&\qquad + \xi_{21}(\psi_1(\xi_1\,;P) - \psi_1(\xi_2\,;P))_x + \xi_{22}(\psi_1(\xi_1\,;P) - \psi_1(\xi_2\,;P))_y\\
&\Delta_h(\psi_2(\xi_1\,;P) - \psi_2(\xi_2\,;P))\\
&\quad = (\xi_{11} - \xi_{21})\,\psi_{2x}(\xi_1\,;P) + (\xi_{12} - \xi_{22})\,\psi_{2y}(\xi_1\,;P)\\
&\qquad + \xi_{21}(\psi_2(\xi_1\,;P) - \psi_2(\xi_2\,;P))_x + \xi_{22}(\psi_1(\xi_1\,;P) - \psi_1(\xi_2\,;P))_y,
\end{aligned} \tag{40}$$

where $\xi_1 = (\xi_{11}, \xi_{12})$, $\xi_2 = (\xi_{21}, \xi_{22})$ and the boundary data is zero. Now looking at (37) we conclude that

$$[\psi_{1x}] + [\psi_{1y}] \leqslant \gamma_1(\eta)\, d^{1-\eta}\{[H_{1x}] + [H_{1y}]\}/(1 - \eta - \nu\gamma_1(\eta)\, d^{1-\eta}), \tag{41}$$

with the same bound holding for $[\psi_{2x}] + [\psi_{2y}]$ but with $[H_{2x}] + [H_{2y}]$ substituted on the right-hand side, whenever we assume that

$$\nu\gamma_1(\eta)\, d^{1-\eta}/(1 - \eta) < 1 \tag{42}$$

for $\eta \in (0, 1)$. Plug (41) into the estimate involving Eq. (40) to obtain that $[\psi(\xi_1\,;P) - \psi(\xi_2\,;P)] \leqslant A_6'[\xi_1(P) - \xi_2(P)]$ where A_6' is obtained in a similar manner as A_6. Hence, if A_6' is less than one we have uniqueness of fixed point solutions to (36) over $\mathcal{O}\!\ell_\nu$. We note that (39), (41) and the size of A_6' may be viewed as constraints on d and the boundary data $z_1(P)$, $z_2(P)$.

In case (42) is not satisfied we may write

$$\begin{aligned}
&\|(\psi_1 - H_1)_{x\bar{x}}\|^2 + 2\,\|(\psi_1 - H_1)_{xy}\|^2 + \|(\psi_1 - H_1)_{y\bar{y}}\|^2\\
&\quad \leqslant 4\nu^2\{\|\psi_{1x}\|^2 + \|\psi_{1y}\|^2\}\\
&\quad \leqslant (2\nu^3\sqrt{\pi}\, d)^2\, \epsilon^{-1} + \epsilon\{\|\psi_{1x\bar{x}}\|^2 + 2\,\|\psi_{1xy}\|^2 + \|\psi_{1y\bar{y}}\|^2\},
\end{aligned}$$

where $\epsilon \in (0, 1)$. Hence, for $\epsilon' \in (0, 1)$,

$$\begin{aligned}
\|\psi_{1x\bar{x}}\|^2 + 2\,\|\psi_{1xy}\|^2 + \|\psi_{y\bar{y}}\|^2 \leqslant (1 - \epsilon' - \epsilon)^{-1}\,\{&(2\nu^3\sqrt{\pi}\, d)^2\, \epsilon^{-1}\\
&+ (\epsilon'^{-1} - 1)(\|H_{1x\bar{x}}\|^2 + 2\,\|H_{1xy}\|^2 + \|H_{y\bar{y}}\|^2)\}.
\end{aligned}$$

Now choose ϵ' and ϵ so that $1 - \epsilon' - \epsilon > 0$. Having obtained this bound we now proceed as in the case of bounding first-order difference quotients for solutions to (18).

We could have defined the set $\mathcal{O}_v$ as $\mathcal{O}_v = \mathcal{O}_{v_1} \times \mathcal{O}_{v_2}$ where

$$\mathcal{O}_{v_i} = \{\xi(P) : \xi(P) = z_i(P), \quad P \in \partial\Omega_h \quad \text{and} \quad [\xi] \leqslant v_i\}.$$

Then (38) would become

$$\begin{aligned} [\psi_1] &\leqslant [H_1] \\ [\psi_2] &\leqslant [H_2]. \end{aligned} \tag{43}$$

This setting would be better if there is a large difference between $[z_1]$, $[z_2]$, $[z_1']$, and $[z_2']$. The analysis would be quite similar.

If we were to replace U_x and U_y, and V_x and V_y by ξ_{1x} and ξ_{1y}, and ξ_{2x} and ξ_{2y}, then the resulting $\psi(\xi(\,\cdot\,); P)$ would not of necessity be a function unless all the quantities ξ_{1x}, ξ_{1y}, ξ_{2x} and ξ_{2y} were nonnegative. This explains our choice of only replacing U and V with ξ_1 and ξ_2.

(IV) *Hölder Continuous Coefficients*

We shall now consider the mildly nonlinear problem with variable coefficients. We seek a solution $U(P)$ to the problem

$$\begin{aligned} a(P)\, U_{x\bar{x}}(P) + c(P)\, U_{y\bar{y}}(P) &= f(P, U(P), U_x(P), U_y(P)), \qquad P \in \Omega_h \\ U(P) &= 0, \qquad P \in \partial\Omega_h, \end{aligned} \tag{44}$$

where we assume $L(f)$ is uniformly bounded, that the coefficients $a(P)$ and $c(P)$ are α-Hoelder continuous over $\bar{\Omega}$ and that $\lambda \leqslant \{a(P), c(P)\} \leqslant L$ for $P \in \Omega$ and for positive numbers λ and L. Let us define the family of mesh functions $\mathcal{O}_1 = \{\xi(P) : \xi(P) = 0,\ P \in \partial\Omega_h\}$. Let $\psi(\xi(\,\cdot\,); P)$ be the solution to the problem, for $\xi(P) \in \mathcal{O}_1$,

$$\begin{aligned} a(P)\, \psi_{x\bar{x}}(\xi(\,\cdot\,); P) + c(P)\, \psi_{y\bar{y}}(\xi(\cdot); P) &= f(P, \xi(P), \xi_x(P), \xi_y(P)), \qquad P \in \Omega_h, \\ \psi(\xi(\,\cdot\,); P) &= 0, \qquad P \in \partial\Omega_h. \end{aligned} \tag{45}$$

Then $[\psi] \leqslant A_1 L L(f)/\lambda$ for each $\xi \in \mathcal{O}_1$. Now define the set of mesh functions $\mathcal{O}_2 = \{\xi(P) : \xi(P) \in \mathcal{O}_1,\ [\xi] \leqslant A_1 L L(f)/\lambda\}$. Then ψ maps $\mathcal{O}_1$ into $\mathcal{O}_2$.

Let $P_0 \in \Omega_h$ and let us rewrite (45) in the form

$$\begin{aligned} &a(P_0)\, \psi_{x\bar{x}} + c(P_0)\, \psi_{y\bar{y}} \\ &\quad = f(P, \xi, \xi_x, \xi_y) + (a(P_0) - a(P))\, \psi_{x\bar{x}} + (c(P_0) - c(P))\, \psi_{y\bar{y}}. \end{aligned} \tag{46}$$

Then we have, for Q running over Ω_h, and $\psi_{x\bar{x}} = \psi_{\bar{x}x}(\xi(\cdot); Q)$, etc.,

$$\begin{aligned} |\psi_x(\xi(\cdot); P_0)| &= \Big| h^2 \sum_Q G_x(P_0; Q)\{f(Q, \xi(Q), \xi_x(Q), \xi_y(Q)) \\ &\qquad + (a(P_0) - a(Q))\, \psi_{x\bar{x}} + (c(P_0) - c(Q))\, \psi_{y\bar{y}}\}\Big| \\ &\leqslant 4\gamma_1(\eta) \Big\{ L(f)\, h^2 \sum_Q \rho_{P_0Q}^{-1-\eta} + L_\alpha(a, c)\, h^2 \\ &\qquad \times \sum_Q \rho_{P_0Q}^{-1-\eta+\alpha}(|\psi_{x\bar{x}}| + |\psi_{y\bar{y}}|)\Big\}. \end{aligned} \tag{47}$$

Using the *Schwarz Inequality* gives

$$h^2 \sum_Q \rho_{P_0Q}^{-1-\eta+\alpha} |\psi_{x\bar{x}}| \leqslant \Big(h^2 \sum_Q \rho_{P_0Q}^{2(\alpha-1-\eta)}\Big)^{1/2} \cdot \Big(h^2 \sum_Q \psi_{x\bar{x}}^2\Big)^{1/2}$$

and

$$h^2 \sum \rho_{P_0Q}^{-1-\eta+\alpha} |\psi_{y\bar{y}}| \leqslant \Big(h^2 \sum \rho_{P_0Q}^{2(\alpha-1-\eta)}\Big)^{1/2} \Big(h^2 \sum \psi_{y\bar{y}}^2\Big)^{1/2}. \tag{48}$$

If $|\alpha - 1 - \eta| < 1$, then the first integrals on the right-hand side of (48) are uniformly bounded. Hence we choose η so that $\alpha > \eta > 0$. From an earlier inequality we have that

$$\|\psi_{x\bar{x}}\| + \|\psi_{y\bar{y}}\| \leqslant 2\sqrt{2}\, L^2 \|f\|/\lambda. \tag{49}$$

Inserting (49) into (48) gives

$$\begin{aligned} [\psi_x(\xi(\cdot); P_0] &\leqslant 4\gamma_1(\eta)\{4L(f)\, d^{1-\eta}/(1-\eta) \\ &\qquad + 2\sqrt{2}\, L_\alpha(a, c)\, L^2 \|f\|\, d^{\alpha-\eta}/\lambda(\alpha - \eta)\}; \end{aligned} \tag{50}$$

the same bound is valid when we consider $[\psi_y(\xi(\cdot); P)]$.

Now define the set $\mathcal{O}\!\ell_2' = \{\xi(P) : \xi(P) \in \mathcal{O}\!\ell_2, [[\xi_x], [\xi_y]] \leqslant$ right-hand side of (50)$\}$. Then ψ maps $\mathcal{O}\!\ell_1$ into $\mathcal{O}\!\ell_2'$. As fixed points of ψ are solutions to (44) we may consider ψ as a mapping from $\mathcal{O}\!\ell_2'$ into itself. As ψ is continuous in ξ the existence of fixed points follows.

We now prove the following theorem.

THEOREM 1.2. *If $f_{,\xi}$, $f_{,p}$ and $f_{,q}$ are uniformly bounded for $P \in \Omega_h$ and for $\xi \in \mathcal{O}\!\ell_2'$ and if $\psi(\xi(\cdot); P)$ is the solution to* (45), *then for $\xi_1(P), \xi_2(P) \in \mathcal{O}\!\ell_2'$ we have*

$$\langle \psi(\xi_1(\cdot); P) - \psi(\xi_2(\cdot); P)\rangle \leqslant A_8 \langle \xi_1(P) - \xi_2(P)\rangle, \tag{51}$$

where $\langle \xi(P) \rangle$ *is defined in* (8) *and*

$$A_8 \leqslant [[f,_\xi], [f,_p], f,_q]] \cdot [A_1/\lambda, 4\gamma_1(\eta)\{d^{1-\eta}/(1-\eta) + 4\sqrt{2\pi}\, L^2 L_\alpha(a, c)\, d^{\alpha-\eta+1}/\lambda(\alpha-\eta)\}]. \tag{52}$$

PROOF. The difference $\psi(\xi_1(\cdot); P) - \psi(\xi_2(\cdot); P)$ solves the problem for $P_0 \in \Omega_h$ and with zero data

$$\begin{aligned} a(P_0)(\psi(\xi_1; P) - \psi(\xi_2; P))_{x\bar{x}} &+ c(P_0)(\psi(\xi_1; P) - \psi(\xi_2; P))_{y\bar{y}} \\ = f(P, \xi_1, \xi_{1x}, \xi_{1y}) &- f(P, \xi_2, \xi_{2x}, \xi_{2y}) \\ &+ (a(P_0) - a(P))(\psi(\xi_1; P) - \psi(\xi_2; P))_{x\bar{x}} \\ &+ (c(P_0) - c(P))(\psi(\xi_1; P) - \psi(\xi_2; P))_{y\bar{y}}. \end{aligned} \tag{53}$$

Hence,

$$[\psi(\xi_1; P) - \psi(\xi_2; P)] \leqslant A_2 F/\lambda$$

where

$$F \equiv [f,_\xi] \cdot [\xi_1 - \xi_2] + [f,_p] \cdot [\xi_{1x} - \xi_{2x}] + [f,_q] \cdot [\xi_{1y} - \xi_{2y}].$$

Applying (49) to (53) gives

$$\begin{aligned} \|(\psi(\xi_1; P) - \psi(\xi_2; P))_{x\bar{x}}\| &+ \|(\psi(\xi_1; P) - \psi(\xi_2; P))_{y\bar{y}}\| \\ &\leqslant 2\sqrt{2}\, L^2 \|f(P, \xi_1, \xi_{1x}, \xi_{1y}) - f(P, \xi_2, \xi_{2x}, \xi_{2y})\|/\lambda \leqslant 2\sqrt{2\pi}\, dFL^2/\lambda. \end{aligned} \tag{54}$$

Substitution of (54) into (47) when applied to (53) gives

$$\begin{aligned} &[(\psi(\xi_1; P) - \psi(\xi_2; P))_x] \\ &\quad \leqslant F\gamma_1(\eta)\{4d^{1-\eta}/(1-\eta) + 4\sqrt{2\pi}\, L^2 L_\alpha(a, c)\, d^{\alpha-\eta+1}/\lambda(\alpha-\eta)\}. \end{aligned}$$

The same inequality is valid for y difference quotients. Our proof is now complete.

(V) *Problems with Nonzero Dirichlet Data*

The results in *Theorem 1.2* remain valid, with a bound similar to that in (52), if we require $U(P) = z(P)$ for $P \in \partial\Omega_h$ where $z(P)$ is continuous on the $\partial\Omega$ and second-order difference quotients of $z(P)$ are uniformly bounded on each side of the $\partial\Omega_h$. In this case we define $\mathfrak{A}_1 = \{\xi(P) : \xi(P) = z(P)$ for $P \in \partial\Omega_h\}$,

$$\mathfrak{A}_2 = \{\xi(P) : \xi(P) \in \mathfrak{A}_1, [\xi(P)] \leqslant A_1 L(f)/\lambda + [z(P)]\},$$

and

$$\mathfrak{A}_2' = \{\xi(P) : \xi(P) \in \mathfrak{A}_2, [[\xi_x], [\xi_y]] \leqslant A_9 + [[H_x], [H_y]]\}$$

where A_9 is the right-hand side of (50) but $\|f\|$ is replaced by

$$\|f\| + 2\sqrt{\pi}\, d[z'']$$

with $[z'']$ the maximum of a second-order difference quotient of $z(P)$ along all sided of the $\partial\Omega_h$ and $H(P)$ is the solution to the problem

$$a(P_0)\, H_{x\bar{x}}(P) + c(P_0)\, H_{y\bar{y}}(P) = 0, \qquad P \in \Omega_h,$$
$$H(P) = z(P), \qquad P \in \partial\Omega_h.$$

Note that bounds of $H_x(P)$ and $H_y(P)$ may be given as $z(P)$ has bounded second-order difference quotients.

(VI) *Bounding Second-Order Difference Quotients*

We now turn our attention to bounding second-order difference quotients for solutions to (44) where we are assuming the same hypotheses hold as in *Theorem 1.2*, that f is α-Hoelder continuous in x and y, and that $U(P)$ is a solution to that problem. Then, for $P_0 \in \Omega_h$,

$$\begin{aligned} U_{x\bar{x}}(P_0) = h^2 \sum_Q G_{x\bar{x}}(P_0\,; Q)\{&f(Q, U(Q), U_x(Q), U_y(Q)) \\ &- f(P_0, U(P_0), U_x(P_0), U_y(P_0))\} \\ &+ h^2 \sum_Q G_{x\bar{x}}(P_0\,; Q) f(P_0, U(P_0), U_x(P_0), U_y(P_0)) \\ &+ h^2 \sum G_{x\bar{x}}(P_0\,; Q)\{(a(P_0) - a(Q))\, U_{x\bar{x}}(Q) \\ &+ (c(P_0) - c(Q))\, U_{y\bar{y}}(Q)\}. \end{aligned} \tag{55}$$

Now

$$\begin{aligned} |f(Q, U, U_x, U_y) &- f(P_0, U, U_x, U_y)| \\ &\leqslant \tilde{A}\{\rho^{\alpha}_{P_0Q} + ([U_x] + [U_y] + U_{x\bar{x}}] + [U_{y\bar{y}}] + [U_{xy}])\, \rho_{P_0Q}\} \end{aligned} \tag{56}$$

where

$$\tilde{A} \leqslant [L_\alpha(f), [f_U], [f_{,p}], [f_{,q}]] \tag{57}$$

and $L_\alpha(f)$ is the Hoelder constant of f with respect to P only. Substitution of (56) into (55) yields the estimate, for $\alpha > \eta > 0$,

$$\begin{aligned} [U_{x\bar{x}}] + [U_{y\bar{y}}] \leqslant 3\gamma_2(\eta)\, &\tilde{A}\{d^{\alpha-\eta}/(\alpha - \eta) + 2d^{1-\eta} A_9 L(f)/(1-\eta)\} \\ &+ 3\gamma_2(\eta)\{\tilde{A} + L_\alpha(a, c)/3\}\, d^{1-\eta} \\ &\times \{[U_{x\bar{x}}] + [U_{y\bar{y}}]\}/(1-\eta) + 3\gamma_5 L(f). \end{aligned} \tag{58}$$

Now if we have that $3\gamma_2(\eta)\{\tilde{A} + L_\alpha(a, c)/3\} d^{1-\eta} < 1 - \eta$, then we have a bound on the sum of all second-order difference quotients of $U(P)$ a fixed point of $\psi(\xi(\cdot); P)$, and these bounds are expressed in terms of known quantities. If $3\gamma_2(\eta)\{\tilde{A} + L_\alpha(a, c)\} d^{1-\eta} \geqslant 1 - \eta$, then we perform an analysis as in [1].

2. Mildly Nonlinear Problems with Bounded Coefficients

In this section we shall consider the following problem: Find the mesh function $U(P)$ such that

$$\begin{aligned} a(P)\, U_{x\bar{x}}(P) + c(P)\, U_{y\bar{y}}(P) &= f(P, U(P), U_x(P), U_y(P)), \qquad P \in \Omega_h\,, \\ U(P) &= 0, \qquad P \in \partial\Omega_h\,, \end{aligned} \tag{1}$$

where we assume that the condition in (2) of Section 1 is satisfied, and that the functions $f_{,\xi}$, $f_{,\xi_x}$ and $f_{,\xi_y}$ are uniformly bounded for $\xi(P) \in \mathcal{O}_1$, and the *only* constraint on $a(P)$ and $c(P)$ is that positive constants λ and L exist so that for very $P \in \Omega$ we have

$$\lambda < \{a(P), c(P)\} < L. \tag{2}$$

The results of this section will remain valid if we require $U(P) = z(P)$ for $P \in \partial\Omega_h$ with $z(P)$ an analytic function on each side of the $\partial\Omega$.

(I) *Bounds on First-Order Difference Quotients*

In this section we shall prove some results which are analogous to some of those in Walsh and Young [7] but for a problem with variable and uniformly bounded coefficients and with analytic Dirichlet data.

Let $H(P)$ be a solution to the problem

$$\begin{aligned} a(P)\, H_{x\bar{x}}(P) + c(P)\, H_{y\bar{y}}(P) &= 0, \qquad P \in \Omega_h\,, \\ H(P) &= x^2 + y^2, \qquad P \in \partial\Omega_h\,, \end{aligned} \tag{3}$$

where $P = (x, y)$ and we are assuming that the condition in (2) holds. Let $P_0 \in \partial\Omega_h$ and let $N(P_0)$ be the set of points in Ω_h which have P_0 as a neighbor; note that for Ω a rectangle we have at most two elements in $N(P_0)$. We define, for any mesh function $W(P)$ having its' domain of definition containing $\bar{\Omega}_h$, a normal difference quotient of $W(P)$ at P_0, denoted by $W_n(P_0)$, by the relation

$$W_n(P_0) = W(P_0) - W(P))/h$$

with $P \in N(P_0)$. The total normal difference quotient, denoted by $W_{\bar{n}}(P_0)$, is defined as

$$W_{\bar{n}}(P_0) = \sum_{P \in N(P_0)} (W(P_0) - W(P))/h.$$

We shall now prove some lemmas without any assumption of regularity for the coefficients $a(P)$ and $c(P)$—not even continuity.

LEMMA 2.1. *If $H(P)$ is the solution to* (3) *and if the coefficients satisfy* (2), *then*

$$\max_{P \in \partial\Omega_h} | H_n(P)| \leqslant [L(\sigma_2 - h)/\lambda, \sigma_1, 2\sigma_1 - h] + [L(\sigma_1 - h)/\lambda, \sigma_2, 2\sigma_2 - h]. \tag{4}$$

PROOF. Let $H^{(1)}(P)$ and $H^{(2)}(P)$ be the solutions to the equation in (3) but now $H^{(1)}(P) = x^2$ for $P \in \partial\Omega_h$ and $H^{(2)}(P) = y^2$ for $P \in \partial\Omega_v$. We shall now bound $H_n^{(1)}(P)$ for $P \in \partial\Omega_h$.

We note that

$$a(P)(H^{(1)}(P) - x^2)_{x\bar{x}} + c(P)(H^{(1)}(P) - x^2)_{y\bar{y}} = -2a(P), \qquad P \in \Omega_h,$$
$$H^{(1)}(P) - x^2 = 0, \qquad P \in \partial\Omega_h.$$

Since the negative of our operator is monotone (Collatz [4, p. 42], then we conclude, using (2), that, for $P \in \bar{\Omega}_h$,

$$H^{(1)}(P) \geqslant x^2.$$

Using the *Maximum Principle* we conclude that $H_n^{(1)}(P) > 0$ for $P \in s_1$ and $H_n^{(1)}(P) < 0$ for $P \in s_3$. Hence, for $P \in s_1$,

$$0 < H_n^{(1)}(P) \leqslant (x^2)_n = 2\sigma_1 - h. \tag{5}$$

Let $K^{(1)}(P) = x^2 + Ly(\sigma_2 - y)/\lambda$. Then

$$\begin{aligned} a(P)(H^{(1)} - K^{(1)})_{x\bar{x}} + c(P)(H^{(1)} - K^{(1)})_{y\bar{y}} & \\ = -2a(P) + 2Lc(P)/\lambda \geqslant 0, & \qquad P \in \Omega_h, \\ H^{(1)}(P) - K^{(1)}(P) = -Ly(\sigma_2 - y)/\lambda \leqslant 0, & \qquad P \in \partial\Omega_h. \end{aligned}$$

Therefore, $K^{(1)}(P) \geqslant H^{(1)}(P)$ for $P \in \bar{\Omega}_h$. Let $P \in s_2$. Then $H^{(1)}(P) = K^{(1)}(P)$ and

$$-L(\sigma_2 - h)/\lambda = K_n^{(1)}(P) \leqslant H_n^{(1)}(P).$$

But $P \in s_2$ gives $H_n^{(1)}(P) \leqslant (x^2)_n = 0$. Therefore, for $P \in s_2$,

$$| H_n^{(1)}(P)| \leqslant L(\sigma_2 - h)/\lambda. \tag{6}$$

In a similar manner, for $P \in s_4$,

$$-L(\sigma_2 - h)/\lambda = K_n^{(1)}(P) \leqslant H_n^{(1)}(P) \leqslant 0. \tag{7}$$

Let $\bar{K}^{(1)}(P) = x^2 + \mu x(\sigma_1 - x)$. Then

$$a(P)(H^{(1)} - \bar{K}^{(1)})_{x\bar{x}} + c(P)(H^{(1)} - \bar{K}^{(1)})_{y\bar{y}} = -2a(P) + 2\mu a(P), \quad P \in \Omega_h,$$
$$H^{(1)}(P) - \bar{K}^{(1)}(P) = -\mu x(\sigma_1 - x), \qquad P \in \partial\Omega_h,$$

and if we choose $\mu \in [1, \infty)$, then $H^{(1)}(P) \leqslant \bar{K}^{(1)}(P)$ for $P \in \bar{\Omega}_h$. Let $\mu = 1$. Then, for $P \in s_3$,

$$-a = \bar{K}_n^{(1)} \leqslant H_n^{(1)}(P) \leqslant 0. \tag{8}$$

Collecting the results in (5), (6), (7), and (8) gives the estimate

$$| H_n^{(1)}(P)| \leqslant [L(\sigma_2 - h)/\lambda, \sigma_1, 2\sigma_1 - h].$$

A similar analysis applies to $H^{(2)}(P)$ and we would define $K^{(2)}(P)$ and $\bar{K}^{(2)}(P)$ by replacing x by y and replacing σ_2 and σ_1 by σ_1 and σ_2.

Our result in (4) now follows from the observation that

$$H(P) = H^{(1)}(P) + H^{(2)}(P).$$

LEMMA 2.2. *If $U(P)$ is the solution to the problem*

$$\begin{aligned} a(P)\, U_{x\bar{x}}(P) + c(P)\, U_{y\bar{y}}(P) &= f(P), \qquad P \in \Omega_h, \\ U(P) &= 0, \qquad P \in \partial\Omega_h, \end{aligned} \tag{9}$$

if $f(P)$ is uniformly bounded for $P \in \Omega_h$, then, for $P \in \partial\Omega_h$,

$$-L'A_{10}/4\lambda \leqslant U_n(P) \leqslant L''A_{10}/4L,$$

where

$$A_{10} = [2\sigma_2 - h, 2\sigma_1 - h, h] + [L(\sigma_2 - h)/\lambda, \sigma_1, 2\sigma_1 - h] + [L(\sigma_1 - h)/\lambda, \sigma_2, 2\sigma_2 - h]. \tag{10}$$

PROOF. Let L' and L'' be positive numbers such that, for each $P \in \Omega_h$,

$$-L' \leqslant f(P) \leqslant L''.$$

Let $R(P) = x^2 + y^2$ where $P = (x, y)$. Then, for $H(P)$ the solution to (1),

$$a(P)(H - R)_{x\bar{x}} + c(P)(H - R)_{y\bar{y}} = -2(a(P) + c(P)), \qquad P \in \Omega_h,$$
$$H(P) - R(P) = 0, \qquad P \in \partial\Omega_h.$$

Let $Z(P) = L'(H(P) - R(P))/4L$ and $Z'(P) = L''(H(P) - R(P))/4\lambda$. Then

$$a(P)(U - Z)_{x\bar{x}} + c(P)(U - Z)_{y\bar{y}} \\ = f(P) + 2L'(a(P) + c(P))/4L, \qquad P \in \Omega_h ,$$

and

$$a(P)(U + Z')_{x\bar{x}} + c(P)(U + Z')_{y\bar{y}} \\ = f(P) - 2L''(a(P) + c(P))/4\lambda, \qquad P \in \Omega_h .$$

Therefore, for each $P \in \bar{\Omega}_h$,

$$-Z'(P) \leqslant U(P) \leqslant Z(P)$$

and hence for each $P \in \partial\Omega_h$,

$$Z_n(P) \leqslant U_n(P) \leqslant Z'_n(P). \tag{11}$$

But

$$| Z'_n(P)| \leqslant L''\{| H_n(P)| + | R_n(P)|\}/4\lambda$$

and $| Z_n(P)| \leqslant L'\{| H_n(P)| + | R_n(P)|\}/4L$, and $| R_n(P)| \leqslant 2\sigma_2 - h$ on s_2 and s_4 , $| R_n(P)| \leqslant 2\sigma_1 - h$ on s_1 and $| R_n(P)| \leqslant h$ on s_3 . Combining these results with those of (4) gives the bound in (10).

The methods of the above theorem may be applied to the more general class of problems which contain lower-order terms. Let $V(P)$ be the solution to the problem

$$a(P)\, V_{x\bar{x}}(P) + c(P)\, V_{y\bar{y}}(P) + g(P)\, V_x(P) + e(P)\, V_y(P) = f(P), \qquad P \in \Omega_h \\ V(P) = 0, \qquad P \in \partial\Omega_h . \tag{12}$$

Now let us define a new function $Z(P)$ by the relation, Bernstein [7, p. 85],

$$V(P) = -[V(P)] - \alpha + \alpha \ln Z(P), \tag{13}$$

where α is a parameter to be determined, ln is the natural logarithm function and $[V(P)] = \max| V(P)|, P \in \bar{\Omega}_h$. If we assume that

$$[| g(P)|, | e(P)|] \leqslant L''', \tag{14}$$

then we may conclude that the quantity $[V(P)]$ is uniformly bounded by an absolute constant. In case $dL''' < 2$ we have

$$[V(P)] \leqslant d^2L[f(P)]/4\lambda(2 - L'''d).$$

The case $dL''' > 2$ may be found in Bers [2, p. 231]. In any event we do know that $V(P)$ is bounded even under the assumption that $f(P) \in \ell_2(\Omega_h)$.

The *Mean Value Theorem* and the relation (13) gives

$$V(P_{01}) - V(P_0) = \alpha(Z(P_{01}) - Z(P_0))/Z(P_0) - 2\alpha(Z(P_{01}) - Z(P_0))^2/Z_1^2(P_0)$$

where $P_0 \in \Omega_h$ and $Z_1(P_0)$ is a value between $Z(P_0)$ and $Z(P_{01})$. Therefore,

$$\begin{aligned}
V_{x\bar{x}}(P_0) &= \alpha Z_{x\bar{x}}(P_0)/Z(P_0) - \alpha Z_x^2(P_0)/Z_1^2(P_0) - \alpha Z_{\bar{x}}^2(P_0)/Z_2^2(P_0),\\
V_{y\bar{y}}(P_0) &= \alpha Z_{y\bar{y}}(P_0)/Z(P_0) - \alpha Z_y^2(P_0)/Z_3^2(P_0) - \alpha Z_{\bar{y}}^2(P_0)/Z_4^2(P_0),\\
V_x(P_0) &= \alpha Z_x(P_0)/Z(P_0) - 2\alpha h Z_x^2(P_0)/Z_1^2(P_0),\\
V_y(P_0) &= \alpha Z_y(P_0)/Z(P_0) - 2\alpha h Z_y^2(P_0)/Z_3^2(P_0),
\end{aligned} \tag{15}$$

where $Z_2(P_0)$ is between $Z(P_0)$ and $Z(P_{03})$, $Z_3(P_0)$ is between $Z(P_0)$ and $Z(P_{02})$ and $Z_4(P_0)$ is between $Z(P_0)$ and $Z(P_{04})$. Therefore substitution of (15) into (12) gives the equation

$$\begin{aligned}
a\alpha\{Z_{x\bar{x}}/Z - Z_x^2/Z_1^2 - Z_{\bar{x}}^2/Z_2^2\} + c\alpha\{Z_{y\bar{y}}/Z - Z_y^2/Z_3^2 - Z_{\bar{y}}^2/Z_4^2\}&\\
+ g\alpha\{Z_x/Z - 2hZ_x^2/Z_1^2\}&\\
+ e\alpha\{Z_y/Z - 2hZ_y^2/Z_3^2\} = f,&
\end{aligned} \tag{16}$$

where the arguments of a, c, g, e, f, $Z_{x\bar{x}}$, $Z_{y\bar{y}}$, Z_x, Z_y, $Z_{\bar{x}}$, $Z_{\bar{y}}$, Z and Z_i, $i = 1,\ldots, 4$, are the arguments of P_0 and P_0 is an arbitrary element of Ω_h. Writing (16) as an equation in $Z(P)$ gives

$$\begin{aligned}
aZ_{x\bar{x}} + cZ_{y\bar{y}} = fZ/\alpha + Z\{(a + 2gh)\, Z_x^2/Z_1^2 + (c + 2eh)\, Z_y^2/Z_3^2&\\
+ aZ_{\bar{x}}^2/Z_2^2 + cZ_{\bar{y}}^2/Z_4^2 - gZ_x/Z - eZ_y/Z\}.&
\end{aligned} \tag{17}$$

Now assume we choose α so that at least $Z(P) \geqslant 0$ over $\bar{\Omega}_h$ and let us consider the expression $(a + 2gh)\, Z_x^2/Z_1^2 - gZ_x/Z$. We choose h so small that $a + 2gh > 0$. Assume that $g \geqslant 0$ and that $Z_x(P_0) \geqslant 0$. Then $Z_1(P_0) \geqslant Z(P_0)$ and hence for an $\epsilon > 0$, $gZ_x/Z = (1 + \epsilon)\, gZ_x/Z_1$. Therefore,

$$(a + 2gh)\, Z_x^2/Z_1^2 - (1 + \epsilon)\, gZ_x/Z_1 \geqslant -d^2(1 + \epsilon)^2/4(a + 2gh). \tag{18}$$

If $d \geqslant 0$ and $Z_x(P_0) \leqslant 0$, then our expression is nonnegative. A similar result applies to the term $(c + 2eh)\, Z_y^2/Z_3^2 - eZ_y/Z$ with

$$(c + 2eh)\, Z_y^2/Z_3^2 - (1 + \epsilon')\, eZ_y/Z \geqslant -e^2(1 + \epsilon')^2/4(c + 2eh), \tag{19}$$

where $\epsilon' > 0$ and h is so small that $c + 2eh > 0$. Therefore the right-hand side of (17) has the lower bound

$$\begin{aligned}
\text{r.h.s. } (17) \geqslant \min f(P) \cdot [Z(P)]/\alpha&\\
- [Z(P)]\{d^2(1 + \epsilon)^2/4(a + 2dh) + e^2(1 + \epsilon')^2/4(c + eh)\};&
\end{aligned} \tag{20}$$

note that ϵ and ϵ' are bounded by $[Z(P)]$. A similar treatment extends to the other cases concerning the sign of g, Z_x, etc.

Let $K(P) = A_{11}(H(P) - R(P))/2\lambda$ where A_{11} is the expression on the right-hand side of (20). Then

$$a(P)(Z-K)_{x\bar{x}} + c(P)(Z-K)_{y\bar{y}} \geqslant 0, \qquad P \in \Omega_h,$$
$$Z(P) - K(P) = \exp\{([V(P)] + \alpha\}, \qquad P \in \partial\Omega_h.$$

Therefore, for $P \in \partial\Omega_h$,

$$Z_n(P) \geqslant K_n(P) \tag{21}$$

and a bound on $K_n(P)$ is given by *Lemma* 2.2. For $P' \in N(P)$ and $P \in \partial\Omega_h$ we have using the *Mean Value Theorem*

$$V(P) - V(P') = \alpha(Z(P) - Z(P'))/Z_1(P) \tag{22}$$

where $Z_1(P)$ is a value between $Z(P)$ and $Z(P')$. Combining (21) and (22) with *Lemma 2.2* gives a lower bound on $V_n(P)$.

To obtain an upper bound on $V_n(P)$ we define the function $Z'(P)$ by the relation

$$V(P) = -[V(P)] - \alpha - \alpha \ln(1 - Z'(P)). \tag{23}$$

Therefore,

$$\begin{aligned}
V_{x\bar{x}}(P_0) &= -\alpha(1-Z')^{-1} Z'_{x\bar{x}} + 2\alpha(1-Z'_1)^{-2} Z'^2_x + 2\alpha(1-Z'_2)^{-2} Z'^2_{\bar{x}},\\
V_{y\bar{y}}(P_0) &= -\alpha(1-Z')^{-1} Z'_{y\bar{y}} + 2\alpha(1-Z'_3)^{-2} Z'^2_y + 2\alpha(1-Z'_4)^{-2} Z'^2_{\bar{y}},\\
V_x(P_0) &= -\alpha(1-Z')^{-1} Z'_x + 2\alpha h(1-Z'_1)^{-2} Z'^2_x, \qquad \text{and}\\
V_y(P_0) &= -\alpha(1-Z')^{-1} Z'_y + 2\alpha h(1-Z'_2)^{-2} Z'^2_y,
\end{aligned}$$

where Z'_i, $i = 1,\ldots, 4$, have similar meaning as earlier, and we obtain the equation

$$\begin{aligned}
aZ_{x\bar{x}} + cZ_{y\bar{y}} = (1-Z')f/\alpha - 2(1-Z')\,\alpha\{&(a+2hg)\,Z'^2_x/(1-Z'_1)^2\\
&+ (c+2he)\,Z'^2_y/(1-Z'_1)^2\\
&+ aZ'^2_{\bar{x}}/(1-Z'_2)^2 + cZ'^2_{\bar{y}}/(1-Z'_4)^2\\
&- gZ'_x/(1-Z') - eZ'_y/(1-Z')\}.
\end{aligned} \tag{24}$$

We choose α so that $0 < 1 - Z < 1$ and perform the same analysis as that used to obtain (20) only now we get a positive constant A_{12} as an upper bound on the right-hand side of Eq. (24). Setting $K'(P) = A_0(H(P) - R(P))/4L$ we get

$$a(P)(Z' + K')_{x\bar{x}} + c(P)(Z' + K')_{y\bar{y}} \leqslant 0, \qquad P_{\bar{x}} \in \Omega_h,$$
$$Z'(P) - K'(P) = 1 - \exp\{([V(P)] + \alpha)/\alpha\}, \qquad P \in \partial\Omega_h.$$

Therefore, for $P \in \partial\Omega_h$,

$$Z_n'(P) \leqslant K_n'(P).$$

We now apply the *Mean Value Theorem* and *Lemma 2.2* to complete the proof of the following lemma.

LEMMA 2.3. *If $V(P)$ is the solution to* (12) *with the conditions* (2) *and* (14) *satisfied, then there exists a positive constant A_7 such that $|V_n(P)| \leqslant A_{13}$ for $P \in \partial\Omega_h$; this constant depends on A_{11}, A_{12} and Ω.*

REMARK 2.1. The effect of the transformations in (13) and (23) is to set our problem in a framework where the lower-order terms are placed in a secondary role. The modifications necessary to include a term of the type $k(P)\, V(P)$, with $k(P) \leqslant 0$, in the proof of *Lemma 2.3* are obvious.

Now we shall consider the problem of bounding first-order difference quotients of $U(P)$. As a first step we shall seek criteria which would imply that first-order difference quotients of $H(P)$ have their maximum and minimum value on the boundary of the domain of definition of the respective first-order difference quotient; we have already obtained bounds there. We now prove the following lemma.

LEMMA 2.4. *If $H(P)$ is the solution to the problem*

$$\begin{aligned} a(P)\, H_{x\bar{x}}(P) + c(P)\, H_{y\bar{y}}(P) = 0, \qquad P \in \Omega_h, \\ H(P) = v(P), \qquad P \in \partial\Omega_h, \end{aligned} \tag{25}$$

if $a(P)$ and $c(P)$ satisfy (2), *then the maximum and the minimum of $H_x(P)$, $H_{\bar{x}}(P)$, $H_{\bar{y}}(P)$, and $H_y(P)$ is on the boundary of their respective domains of definition.*

PROOF. Let $P \in \Omega_h$ with P_1, P_2, P_3 and P_4 the neighbors of P. Using the result that

$$a(P_1)(H_{x\bar{x}}(P_1) - H_{x\bar{x}}(P)) + c(P_1)(H_{y\bar{y}}(P_1) - H_{y\bar{y}}(P))$$
$$+ (a(P_1) - a(P))\, H_{x\bar{x}}(P) + (c(P_1) - c(P))\, H_{y\bar{y}}(P) = 0$$

and using the fact that $H_{y\bar{y}}(P) = -a(P)\, H_{x\bar{x}}(P)/c(P)$ gives the relation

$$\begin{aligned}\{a(P_1) + c(P_1)(2 + a(P)/c(P))\}\, H_x(P) \\ = \{c(P_1)\, a(P)/c(P)\}\, H_x(P_3) \\ + a(P_1)\, H_x(P_1) + c(P_1)\, H_x(P_2) + c(P_1)\, H_x(P_4).\end{aligned} \tag{26}$$

Now, on account of the condition in (2), all the coefficients of ther terms on the right-hand side of (26) are positive. Since the sum of the coefficients on the right-hand side is equal to the coefficient on the left-hand side we have $H_x(P)$ expressed as a linear combination—with positive coefficients having a sum equal to one— of $H_x(P_i)$, $t = 1,..., 4$. The above result requires P_i, $i = 1,..., 4$, to be in the set $\bar{\Omega}_{hx}$.

Let $P \in \Omega_h$ and P_i, $i = 1,..., 4$, be elements of $\bar{\Omega}_{hy}$. Then we obtain

$$\begin{aligned}a(P_2)(H_{x\bar{x}}(P_2) - H_{x\bar{x}}(P)) + c(P_2)(H_{y\bar{y}}(P_2) - H_{y\bar{y}}(P)) \\ + (a(P_2) - a(P))\, H_{x\bar{x}}(P) + (c(P_2) - c(P))\, H_{y\bar{y}}(P) = 0,\end{aligned}$$

and now using the fact that $H_{x\bar{x}}(P) = -a(P)\, H_{y\bar{y}}(P)/c(P)$ we have the relation

$$\begin{aligned}\{c(P_2) + a(P_2)(2 + c(P)/a(P))\}\, H_y(P) \\ = \{a(P_2)\, c(P)/a(P)\}\, H_y(P_4) \\ + a(P_2)\, H_y(P_2) + c(P_1)\, H_y(P_3) + c(P_1)\, H_y(P_1).\end{aligned}$$

Again $H_y(P)$ is a linear combination of $H_y(P_i)$, $i = 1,..., 4$, and our result now follows.

A similar treatment extends to $H_{\bar{x}}$ and $H_{\bar{y}}$.

REMARK 2.2. From *Lemma 2.4* and the definition of $H^{(1)}(P)$ in *Lemma 2.1* we conclude that $H_x^{(1)}(P) \geqslant 0$ for all $P \in \Omega_h$.

As an immediate consequence of the above lemmas we have the following theorem.

THEOREM 2.1. *If $U(P)$ satisfies (9) with $f(P)$ identically constant over Ω_h and if $a(P)$ and $c(P)$ satisfy condition (2) over $\bar{\Omega}$, then $U_x(P)$, $U_{\bar{x}}(P)$, $U_y(P)$ and $U_{\bar{y}}(P)$ are uniformly bounded for $P \in \Omega_h$.*

The reason we have restricted *Theorem* 2.1 to the case that $f(P)$ is identically constant is that the case of continuously differentiable $f(P)$ would require that we bound $h^2 \sum_{Q \in \Omega_h} G(P; Q)$ where $G(P; Q)$ is the solution to the problem, for $Q \in \Omega_h$,

$$\begin{aligned}a(P_1)\, G_{x\bar{x}}(P; Q) + c(P_1)\, G_{y\bar{y}}(P; Q) \\ + (a_x(P) - c_x(P)\, a(P)/c(P))\, G_x(P; Q) = -\delta(P; Q)/h^2, \qquad P \in \Omega_h, \\ G(P; Q) = 0, \qquad P \in \partial\Omega_h,\end{aligned}$$

with all difference quotients taken with respect to the components of P. Such a bound appears difficult unless we assume smoothness of $a(P)$ and $c(P)$; however in that case the method presented here is less general than the development in McAllister [1].

Now let us consider a generalization of *Lemma 2.2.*

LEMMA 2.5. *If $H(P)$ is the solution of the equation in* (1) *but now $H(P) = x^m$ for $P \in \partial\Omega_h$ or $H(P) = y^m$ for $P \in \partial\Omega_h$ where m is a positive integer, then, for $P \in \partial\Omega_h$,*

$$| H_n(P)| \leqslant \Bigg[\sum_{j=0}^{m-1} \sigma_1^{m-1-j} h^j, L(\sigma_2 - h) \sum_{j=0}^{m-2} \sigma_2^{m-2-j} \times (\sigma_2 - h)^j/\lambda, L(\sigma_2^{m-1} - h^{m-1})/\lambda, \sigma_1^{m-1} \Bigg]. \tag{27}$$

PROOF. We consider the case that $H(P) = x^m$ for $P \in \partial\Omega_h$. Then we conclude that $H(P) \geqslant x^m$ and hence for $P \in s_1$, $H_n(P) > 0$ and for $P \in s_3$, $H_n(P) < 0$. Hence for $P \in s_1$,

$$0 < H_n(P) \leqslant (x^m)_n = \sum_{j=0}^{m-1} \sigma_1^{m-1-j} h^j.$$

For $P \in s_2$, we have that $H_n(P) \leqslant 0$, $H(P) \leqslant Ly(\sigma_2^{m-1} - y^{m-1})/\lambda + x^m$, and

$$-L\lambda^{-1}(\sigma_2 - h) \sum_{j=0}^{m-2} \sigma_2^{m-2-j}(\sigma_2 - h)^j$$
$$= (Ly(\sigma_2^{m-1} - y^{m-1})/\lambda + x^m)_n \leqslant H_n(P) \leqslant 0.$$

For $P \in s_4$ we have $0 \geqslant H_n(P) \geqslant -L(\sigma_2^{m-1} - h^{m-1})/\lambda$. For $P \in s_3$, the function $x^m + x(\sigma_1^{m-1} - x^{m-1}) \geqslant H(P)$ and

$$-(\sigma_1^{m-1} - h^{m-1}) - h^{m-1} = (x^m + x(\sigma_1^{m-1} - x^{m-1}))_n \leqslant H_n(P) \leqslant 0.$$

Note that

$$(x + h)^m + (x - h)^m - 2x^m = 2h^2 \sum_{j=0}^{m-1} x^{m-1-j} \sum_{i=0}^{j-1} (x + h)^i (x - h)^{j-1-i}.$$

If $H(P)$ is the solution to (25), then we may assume that the boundary data $v(P)$, which is assumed continuous on the $\partial\Omega$, is zero on all vertices; this may be accomplished by subtracting a plane of the form $B_1x + B_2xy + B_3y + B_4$, which itself solves Eq. (1), from $v(P)$. Suppose that $v(P)$ is an analytic function on s_i, $i = 1,\ldots, 4$, which absolutely converges on a closed interval containing each side s_i; note that on each side $v(P)$ is a function of only one variable. Let $H^{(j)}(P)$, $j = 1,\ldots, 4$, be the solution to Eq. (1) such that $H^{(j)}(P) = v(P)$ for $P \in s_j$ and $H^j(P) = 0$ for $P \in s_i$ for $i \neq j$. Then for the power series expansion of $v(P)$ on s_j we have by *Lemma 2.5* that $|H_n^{(j)}(P)|$ is uniformly bounded for $P \in \partial\Omega_h$. Now observing that $H(P) = \sum_{j=1}^4 H^{(j)}(P)$ we conclude with the following theorem.

THEOREM 2.2. *If $H(P)$ is the solution to* (25) *with analytic data, then, for each $P \in \partial\Omega_h$, $|H_n(P)|$ is uniformly bounded by a constant which is uniform in h; hence we also have a uniform bound on all first-order difference quotients of $H(P)$.*

Let S be the disc of radius r with center at the origin. Let S_h be those mesh points in S such that each neighbor is in $\bar{S}$ and let ∂S_h be those mesh points with at least one neighbor in the exterior of $\bar{S}$. Then the function $R(P) = (r^2 - x^2 - y^2)/2$ solves the equation

$$a(P)\, R_{x\bar{x}}(P) + c(P)\, R_{y\bar{y}}(P) = -(a(P) + c(P))$$

for $P \in S_h$ and $|R(P)| \leqslant h \max_{P\in S_h} |x + y - h|$ for $P \in \partial S_h$. Hence the function $W(P)$ which solves the problem

$$\begin{aligned} a(P)\, W_{x\bar{x}}(P) + c(P)\, W_{y\bar{y}}(P) &= -(a(P) + c(P)), \qquad && P \in S_h, \\ W(P) &= 0, && P \in \partial S_h, \end{aligned}$$

in such that $[W(P) - R(P)] \leqslant h \max_{P\in S_h} |x + y - h|$. Therefore,

$$|W_n(P)| \leqslant 2 \max_{P\in S_h} |x + y - h|. \tag{28}$$

The estimate in (28) may now be used to prove the following extension of *Theorem 2.1* to the case that our domain is a disc of radius r.

THEOREM 2.3. *If $U(P)$ solves* (9) *over S_h and if $a(P)$ and $c(P)$ satisfy* (2), *and $f(P)$ is bounded on S, then all first-order difference quotients of $U(P)$ are bounded over ∂S_h. If* (2) *is satisfied and $f(P)$ is identically constant, then first-order difference quotients of $U(P)$ are bounded.*

(II) *Application of* a Priori *Bounds to Mildly Nonlinear Problems*

Let $\mathcal{O}\!\ell_2 = \{\xi(P) : \xi(P) \in \mathcal{O}\!\ell_1$ and $[\xi(P)] \leqslant Ld^2L(f)/8\lambda\}$ with $\mathcal{O}\!\ell_1$ defined in Section 1. Let $\psi(\xi(\cdot); P)$ be the solution to the problem, for $\xi(P) \in \mathcal{O}\!\ell_1$,

$$\begin{aligned} a(P)\, \psi_{x\bar{x}}(\xi(\cdot); P) + c(P)\, \psi_{y\bar{y}}(\xi(\cdot); P) &= f(P, \xi(P), \psi_x(P), \psi_y(P)), \quad P \in \Omega_h, \\ \psi(\xi(\cdot); P) &= 0, \qquad P \in \partial\Omega_h. \end{aligned} \tag{29}$$

Then $\psi : \mathfrak{A}_1 \to \mathfrak{A}_2$. From the estimate in (10) of Section 2 we have that, for any $\xi(P) \in \mathfrak{A}_1$,

$$\max_{P \in \partial\Omega_h} \{| \psi_n(\xi(\cdot); P)|\} \leqslant A_{10}[f]/4\lambda. \tag{30}$$

Let $\mathfrak{A}_2' = \{\xi(P) : \xi(P) \in \mathfrak{A}_2$ and $\max_{P \in \partial\Omega} \{| \xi_n(P)| \leqslant A_{10}[f]/4\lambda\}$. Then $\psi : \mathfrak{A}_1 \to \mathfrak{A}_2'$ and any solution to our problem in this section is an $\xi_0(P) \in \mathfrak{A}_2'$ such that $\psi(\xi_0(\cdot); P) = \xi_0(P)$.

Let $\xi_1(P), \xi_2(P) \in \mathfrak{A}_2'$. Then, for $P \in \Omega_h$,

$$\begin{aligned} &a(P)(\psi(\xi_1(\cdot); P) - \psi(\xi(\cdot); P))_{x\bar{x}} + c(P)(\psi(\xi(\cdot); P) - \psi(\xi(\cdot); P))_{y\bar{y}} \\ &\quad = f(P, \xi_1(P), \psi_{1x}(P), \psi_{1y}(P)) - f(P, \xi_2(P), \psi_{2x}(P), \psi_{2y}(P)) \end{aligned} \tag{31}$$

with zero Dirichlet data. Therefore,

$$[\psi(\xi_1(\cdot); P) - \psi(\xi_2(\cdot); P)] \leqslant A_3'[\xi_1(P) - \xi_2(P)], \tag{32}$$

where

$$A_3' \leqslant Ld^2 \max_{\xi\epsilon} \{| \partial f/\partial\xi |\}/8\lambda$$

whenever $h \max\{| \partial f/\partial\xi_x |, | \partial f/\partial\xi_y |\} \leqslant 1$.

We have proved the following theorem.

THEOREM 2.4. *If $A_3' < 1$, then a unique solution to the problem in* (1) *exists and may be obtained iteratively with any initial choice and with convergence the order of a geometric series.*

3. On Some Fundamental Inequalities

We shall prove here some inequalities we have used in the development of this paper.

Let $U(P)$ be defined on Ω_h with $U(P) = 0$ on the $\partial\Omega_h$. Let $P_0 \in \Omega_h$. Then

$$U(P_0) = h \sum_{\ell_x(P_0)} U_{(x)}(P), \qquad U(P_0) = h \sum_{\ell_y(P_0)} U_{(y)}(P), \tag{1}$$

where $\ell_x(P_0)$ or $\ell_y(P_0)$ is a line segment from P_0 to the $\partial\Omega_h$ such that it is parallel to the x-axis or the y-axis, and $U_{(x)}$ and $U_{(y)}$ is a difference quotient which makes (1) true, i.e., $U_x(P)$ or $U_{\bar{x}}(P)$ and $U_y(P)$ or $U_{\bar{y}}(P)$. Applying the *Schwarz Inequality* to (1) gives

$$U^2(P_0) \leqslant \sigma_1 h \sum U_x^2(P),$$

and

$$U^2(P_0) \leqslant \sigma_2 h \sum U_y^2(P), \tag{2}$$

where the sums in (2) are over the line thru P_0 which is parallel to the x-axis or the y-axis, respectively. Now sum (2) as $P_0 \in \Omega_h$ to get

$$h^2 \sum_{\Omega_h} U^2(P) \leqslant \tfrac{1}{2}\sigma_1 \cdot \sigma_2 h^2 \sum_{\Omega_h} \{U_x{}^2 + U_y{}^2\}. \tag{3}$$

Hence we have proved the *Poincaré Inequality* which is given in (3).

Let $U(P)$ be the solution to the uniformly elliptic problem

$$\begin{aligned} a(P)\, U_{x\bar{x}}(P) + 2b(P)\, U_{xy} + c(P)\, U_{\bar{y}y}(P) = f(P), \qquad & P \in \Omega_h\,, \\ U(P) = 0, \qquad & P \in \partial\Omega_h\,. \end{aligned} \tag{4}$$

Then we claim that

$$\| U_{xx} \|^2 + 2\| U_{xy} \|^2 + \| U_{yy} \|^2 \leqslant 2L^2 \| f \|^2 / \lambda^4; \tag{5}$$

this inequality was proved in [3] but for a unit square. To prove (5) we need only show that

$$h^2 \sum_{\Omega_h} U_{xy}^2 = h^2 \sum_{\Omega_h} U_{x\bar{x}} U_{y\bar{y}}\,, \tag{6}$$

where

$$U_{xy}(x, y) = \{U(x + h, y + h) - U(x, y + h) - U(x + h, y) + U(x, y)\}/h^2.$$

Now extend $U(P)$ to the lines $y = \sigma_2 + h$ and $y = -h$ by defining it zero there. Then the discrete form of *Gauss's Theorem* gives

$$\begin{aligned} h^2 \sum_{\Omega'_h} U_{xx} U_{yy} &= -h^2 \sum_{\Omega'_h} U_{xxy} U_y + h \sum_{\partial\Omega'_h} U_{xx} U_y \\ \text{and} \qquad h^2 \sum_{\Omega'_h} U_{xy} U_{xy} &= -h^2 \sum_{\Omega'_h} U_{xyx} U_y + h \sum_{\partial\Omega'_h} U_{xy} U_y\,, \end{aligned} \tag{7}$$

where in this instance $\bar{\Omega}'_h = \bar{\Omega}_h \cup s'_4$ and s'_4 is the intersection of our grid with $y = -h$. Our results follow once we observe that $U_y = 0$ on s_1 and s_3, $U_{xx} = 0$ on s_2 and s_4 and $U_{xy} = 0$ on s_2 and s'_4. Now note that s'_4 contributes nothing to the left-hand side of (7). Our proof of (5) now follows that of [3].

We shall now prove, for rectangular regions with square mesh, the discrete form of the *Sobolev Inequality*. Let $W(P)$ be a mesh function having $W(P) = 0$ for $P \in \partial\Omega_h$. Let κ_1 and κ_2 be the smallest positive integers such that

$$\kappa_1 \sigma_1 = \kappa_2 \sigma_2\,;$$

such integers exist as we are assuming σ_1 and σ_2 are rational numbers. Now reflect $W(P)$ about the line $x = \sigma_1$ for κ_1 times and reflect $W(P)$ about the line $y = \sigma_2$ for κ_2 times. Now fill out $W(P)$ so that it is defined over a square with sides $\kappa_1\sigma_1$; note that $W(P) = 0$ on the boundary of this square. Let Ω_1 be the unit square and let Ω_{1h} be the intersection of our grid system with Ω_1. Let $P \in \Omega_{1h}$ and define $W(P) = W(P')$ where $P' = (x_P/\kappa_1\sigma_1, y_P/\kappa_1\sigma_1)$ where $P = (x_P, y_P)$. Then using [3, p. 304]

$$[W(P)] \leqslant \sqrt{1+\pi}\,\{\| W_{x\bar{x}}(P)\| + \| W_{y\bar{y}}(P)\|\}/4 \tag{8}$$

where the norms are over Ω_{1h}. Now if Ω_h' denotes the mesh region associated with the square of sides $\kappa_1\sigma_1$, then

$$[W(P)] \leqslant \kappa_1^2 \cdot \kappa_1^2 \cdot \sigma_1 \cdot \sigma_2\,\{\sqrt{1+\pi}/4\} \cdot \{\| W_{x\bar{x}}\| + \| W_{y\bar{y}}\|\} \tag{9}$$

where the norms in (9) are over Ω_h and $\kappa_1 \cdot \kappa_2$ is the total number of reflections of Ω_h to make-up the region Ω_h'.

References

1. G. T. McAllister. A Priori bounds on difference quotients of solutions to some linear uniformly elliptic difference equations. *Numer. Math.* **11** (1968).
2. L. Bers. On mildly nonlinear partial differential equations of elliptic type. *J. Res. Nat. Bur. Standards* **51** (1953).
3. J. Nitsche and J. C. C. Nitsche. Error estimates for the numerical solution of elliptic differential equations. *Arch. Rational Mech. Anal.* **5** (1960).
4. L. Collatz. "The Numerical Treatment of Differential Equations." Springer-Verlag, Berlin, 1960.
5. H. L. Smolickii. Estimates of the derivatives of fundamental functions. *Dokl. Akad. Nauk. SSSR(NS)* **74** (1950), 205.
6. J. L. Walsh and D. Young. Lipschitz conditions for harmonic and discrete harmonic functions. *J. Math. Phys.* **36** (1957), 138–150.
7. S. Bernstein. Sur la généralisation du problème de Dirichlet II. *Math. Ann.* **69** (1910).